Eigenvalue Analysis of Neuronal Firing Stability in a Two-Dimensional Linear Dynamical System

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Abstract

We investigate the stability of simplified neuronal dynamics through the spectral analysis of two-dimensional linear dynamical systems of the form $d\mathbf{X}/dt = A\mathbf{X}$, where $\mathbf{X}$ encodes membrane potential and a recovery variable. By systematically varying the system matrix A and computing its eigenvalues, we identify three qualitatively distinct behavioural regimes: stable decay, damped oscillation resembling biological firing patterns, and unstable divergence. We derive analytical conditions for each regime in terms of the trace and determinant of A, validate these conditions through numerical simulation, and visualize the resulting phase portraits and temporal trajectories. A trace-determinant stability map provides an intuitive geometric summary of the parameter space. Our results demonstrate that eigenvalue structure is a powerful and interpretable predictor of neuron-like dynamical behaviour, offering a transparent linear-algebraic foundation for more complex computational neuroscience models.
Keywords: eigenvalue analysis, linear dynamical systems, computational neuroscience, stability theory, membrane potential, phase portrait

1 Introduction

The dynamics of neuronal firing lie at the intersection of biology, mathematics, and engineering. Understanding how neurons transition between resting, oscillatory, and unstable states is fundamental to both neuroscience and the design of neuromorphic computing architectures. While fully realistic models such as the Hodgkin-Huxley system capture the richness of biological detail, their nonlinearity makes analytical tractability difficult, especially for newcomers to the field.

Linear dynamical systems offer a complementary perspective. By restricting attention to the linearised regime near an equilibrium, we can invoke the full machinery of linear algebra—eigenvalues, eigenvectors, and the spectral mapping theorem—to characterise long-run behaviour exactly. The central object of interest is the system matrix A: its eigenvalues determine whether trajectories converge, oscillate, or diverge, providing a direct algebraic handle on stability.

This paper examines a minimal two-variable model inspired by the leaky integrate-and-fire (LIF) neuron and its extensions. We treat the membrane potential V and a recovery variable W as the state vector, and study how varying the four entries of A produces qualitatively different dynamics. The contribution is deliberately elementary: rather than novelty in modelling, we aim for depth in analysis, clarity in visualisation, and pedagogical value. The code, figures, and LaTeX source are publicly available at `https://github.com/calyxish/neuronal-stability-analysis`.

2 Mathematical Framework

2.1 The Two-Dimensional Neuronal Model

We consider the autonomous linear system

$$\frac{d\mathbf{X}}{dt} = A\mathbf{X}, \quad \mathbf{X}(0) = \mathbf{X}_0 \tag{1}$$

where $\mathbf{X} = [V, W]^\top$, V denotes membrane potential, W is a recovery (adaptation) variable, and

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{2}$$

This formulation is motivated by the linearisation of the FitzHugh-Nagumo model about its rest state. The LIF neuron corresponds to the scalar case $W = 0$, $b = c = 0$, $a = -1/\tau$, which yields the single eigenvalue $\lambda = -1/\tau < 0$, confirming intrinsic stability.

2.2 Eigenvalue Analysis

The qualitative behaviour of (1) is governed by the eigenvalues of A, computed from the characteristic polynomial

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0 \tag{3}$$

with solutions

$$\lambda_{1,2} = \frac{\mathrm{tr}(A) \pm \sqrt{\mathrm{tr}(A)^2 - 4\det(A)}}{2}. \tag{4}$$

Three canonical cases arise depending on the discriminant $\Delta = \mathrm{tr}(A)^2 - 4\det(A)$:

Case I: Real eigenvalues ($\Delta > 0$). The system exhibits exponential growth or decay along two independent directions. Stability requires both eigenvalues negative, i.e., $\mathrm{tr}(A) < 0$ and $\det(A) > 0$.

Case II: Complex eigenvalues ($\Delta < 0$). Writing $\lambda = \alpha \pm i\beta$, the solution oscillates at angular frequency β while the amplitude grows or decays at rate $e^{\alpha t}$. This case produces the spiral trajectories most reminiscent of neuronal firing cycles.

Case III: Pure imaginary eigenvalues ($\mathrm{tr}(A) = 0$, $\det(A) > 0$). The system is a centre: closed, periodic orbits with no net amplitude change. Biologically, this corresponds to an idealized, perfectly sustained oscillation.

2.3 Stability Conditions

Combining the above, the complete stability classification is summarised in Table 1.

Table 1: Classification of dynamical regimes by trace-determinant conditions. $\Delta = \mathrm{tr}(A)^2 - 4\det(A)$ is the discriminant.

Condition	**Eigenvalue Type**	**Neuronal Behaviour**
tr < 0, det > 0, $\Delta > 0$	Real, negative	Decay to rest (Stable node)
tr < 0, det > 0, $\Delta < 0$	Complex, Re < 0	Damped oscillation (Neuron-like firing)
tr $= 0$, det > 0	Pure imaginary	Sustained oscillation (Centre)
tr > 0, det > 0, $\Delta < 0$	Complex, Re > 0	Unstable spiral (Runaway firing)
det < 0	Real, mixed sign	Unstable (Saddle point)

3 Methods

3.1 Numerical Integration

Trajectories were computed using the forward Euler scheme

$$\mathbf{X}_{k+1} = \mathbf{X}_k + dt\, A\mathbf{X}_k, \quad dt = 0.005\,\mathrm{s}. \tag{5}$$

This first-order explicit method is sufficient for qualitative analysis given the moderate time horizons ($T \leq 10$ s) and relatively small spectral radii considered. All simulations were initialised from $\mathbf{X}_0 = [1, 0]^\top$.

3.2 Parameter Configurations

Four system matrices were studied in detail:

- $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ (Stable Node)
- $A_2 = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$ (Stable Oscillatory)
- $A_3 = \begin{bmatrix} 0.5 & 2 \\ -2 & 0.5 \end{bmatrix}$ (Unstable Oscillatory)
- $A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ (Unstable Node)

These were chosen to span the principal stability regions of the trace-determinant plane.

3.3 Stability Map

A 600×600 grid over $\mathrm{tr} \in [-3, 3]$ and $\det \in [-1, 5]$ was classified pixel-by-pixel using the analytic conditions from Section 2. The three numerical examples were overlaid as markers to validate the classification.

4 Results

4.1 Phase Portraits

Figure 1 presents the phase portraits for all four parameter configurations. The stable node (A_1) produces two independent exponential decay channels, and all trajectories converge monotonically to the origin. The stable oscillatory case (A_2) yields an inward spiral: both the voltage V and recovery variable W oscillate while their amplitudes decay, closely mimicking a sub-threshold neuronal ping. The unstable configurations (A_3, A_4) produce outward spirals and rays respectively, indicating runaway dynamics inconsistent with physiological firing without a reset mechanism.

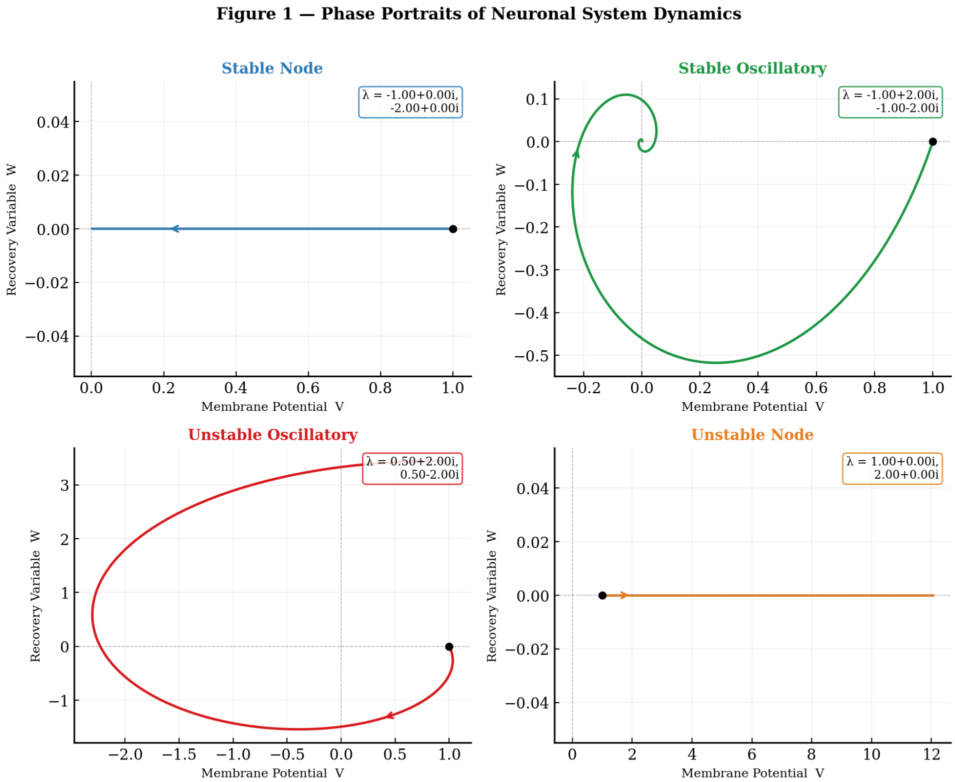

Figure 1: Phase portraits of the two-dimensional neuronal system under four parameter regimes. Arrows indicate the direction of increasing time. Eigenvalues are displayed in each panel.

4.2 Temporal Evolution

Figure 2 shows the time series of both state variables. In the stable oscillatory regime, the membrane potential exhibits decaying sinusoidal oscillations consistent with damped sub-threshold dynamics observed experimentally. The period of oscillation is $T_{\text{osc}} = 2\pi/|\operatorname{Im}(\lambda)| = \pi$ s, confirming that the imaginary part of the eigenvalue directly sets the firing-like frequency.

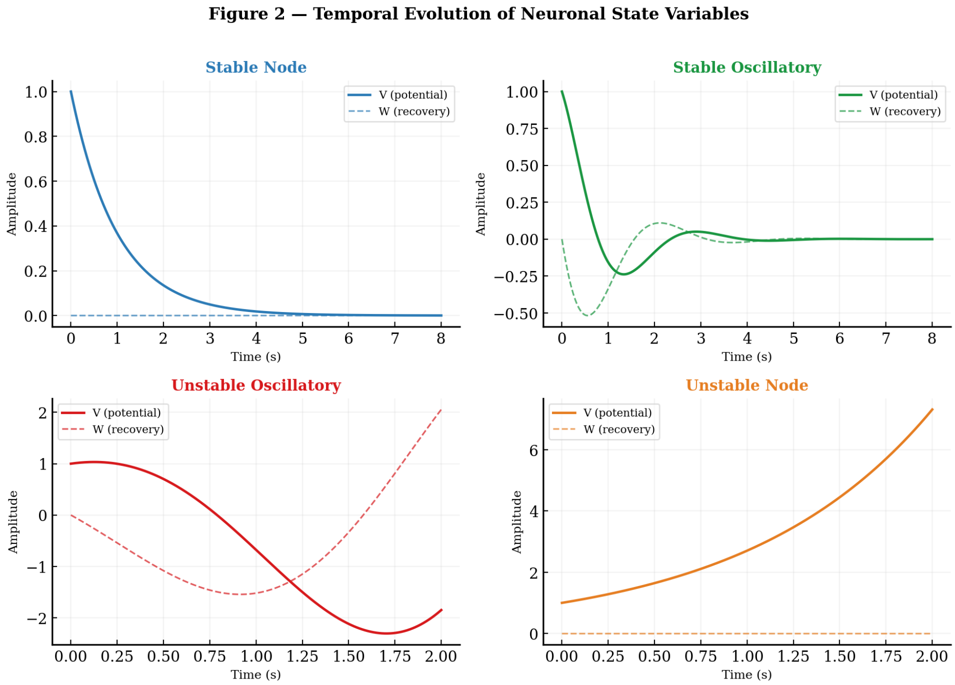

Figure 2: Temporal evolution of V (solid) and W (dashed) for each system configuration. Oscillatory structure is visible in configurations A_2 and A_3.

4.3 Trace-Determinant Stability Map

Figure 3 visualises the full parameter space. The parabola $\Delta = 0$ separates spiral from nodal regions; the vertical axis $\mathrm{tr} = 0$ separates stable from unstable halves. The negative determinant half-plane corresponds universally to saddle-point (unstable) behaviour regardless of trace. The three numerical examples fall precisely in the predicted coloured regions, confirming the analytic classification.

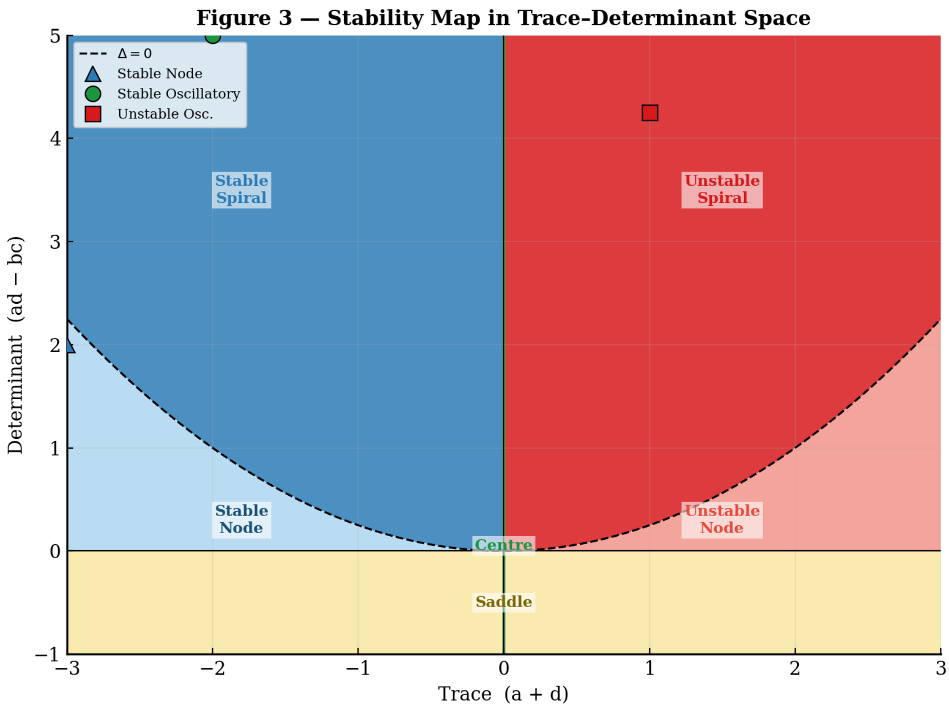

Figure 3: Stability map in trace-determinant space. Coloured regions correspond to distinct dynamical regimes. Markers denote the three studied configurations; the dashed parabola is the discriminant boundary $\Delta = 0$.

4.4 Eigenvalue Locus

Figure 4 traces the locus of eigenvalues in the complex plane as the diagonal parameter a is varied from -2.5 to 1.5, with off-diagonal entries held at $b = 2$, $c = -2$. The locus crosses the imaginary axis (vertical dashed line) at exactly $a = 0$, the bifurcation point beyond which the system becomes unstable. This provides a continuous view of the stable-to-unstable transition analogous to the Hopf bifurcation in nonlinear systems.

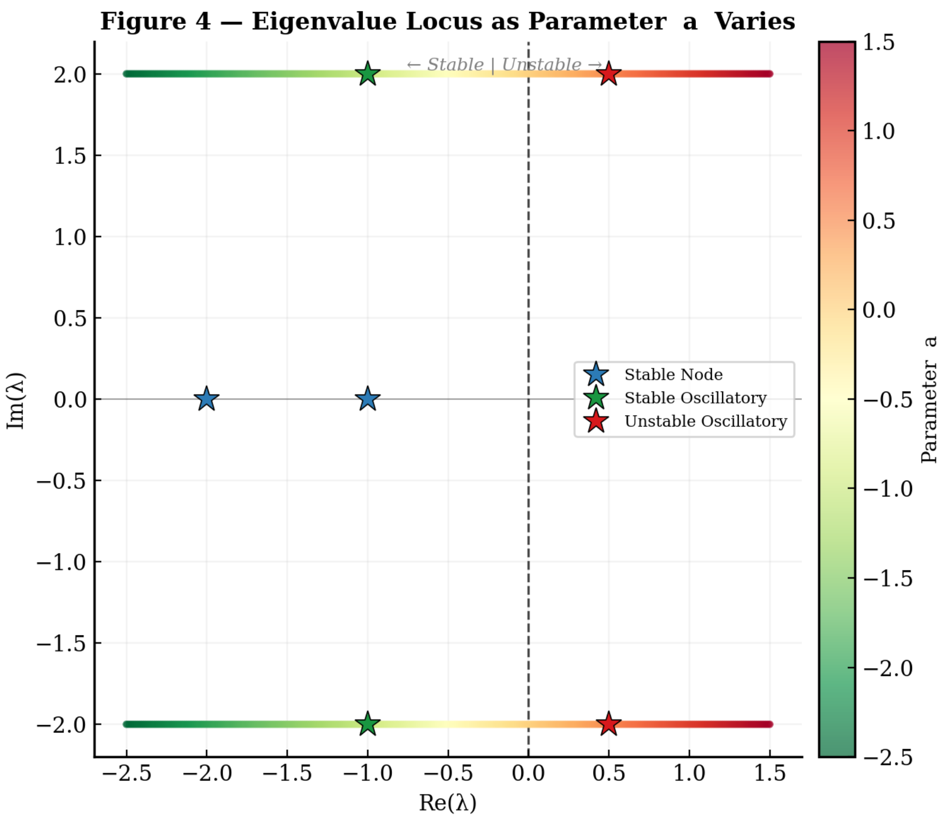

Figure 4: Eigenvalue locus in the complex plane as parameter a varies from -2.5 (green) to 1.5 (red). Stars mark eigenvalues for the three studied matrices. The vertical dashed line $\text{Re}(\lambda) = 0$ separates stable from unstable regimes.

5 Discussion

The principal finding of this study is that the qualitative behaviour of a simplified neuronal system—stable decay, oscillatory firing, or unstable divergence—is fully determined by two scalar quantities: the trace and the determinant of the system matrix. This is a striking example of how algebraic structure encodes physical behaviour.

The damped oscillatory regime is of particular biological interest. Neurons in vivo often exhibit sub-threshold oscillations before reaching a firing threshold, and the linear model captures this qualitatively through complex eigenvalues with negative real parts. The imaginary component sets the intrinsic resonance frequency, which in real neurons influences the preferred input frequency—a phenomenon known as resonance.

From a dynamical systems perspective, the transition from stable spiral to unstable spiral as $\text{tr}(A)$ passes through zero is the linear analogue of a Hopf bifurcation. In nonlinear systems, this transition gives rise to limit cycles—the periodic orbits that model repetitive action potential firing. The present analysis therefore provides the correct intuition for understanding why nonlinear neuron models can produce sustained oscillations without external drive.

The eigenvalue locus in Figure 4 makes the bifurcation mechanism visually explicit: as a single parameter increases, the eigenvalue pair migrates from the stable left half-plane across the imaginary axis into the unstable right half-plane. This kind of continuous parameter sensitivity analysis—simple to implement in the linear setting—generalises to local bifurcation analysis in nonlinear models via the Jacobian linearisation.

A practical implication concerns the design of neuromorphic circuits. If a designer wishes to engineer oscillatory dynamics without instability, the target operating point lies in the stable spiral region of the trace-determinant plane. The stability map (Figure 3) provides an immediate geometric guide for parameter selection.

6 Limitations and Future Work

This study is subject to several important limitations. First, linearity is a strong assumption. Biological neurons are intrinsically nonlinear: they exhibit thresholding, spike generation via sodium channel activation, and refractory periods. The linear model studied here does not capture action potential initiation and is therefore valid only in the sub-threshold regime near equilibrium.

Second, dimensionality is restricted to two. Real neural populations involve high-dimensional, coupled dynamics where the eigenspectrum of large matrices governs collective behaviour. Extending the analysis to higher-dimensional systems and sparse connectivity matrices is a natural next step.

Third, the numerical integration uses forward Euler, which is first-order accurate and conditionally stable. A Runge-Kutta scheme would improve accuracy for stiff configurations, though qualitative conclusions would be unchanged.

Fourth, no comparison with experimental data is attempted. Future work could calibrate the model against patch-clamp recordings and assess whether the linearised system captures observed oscillation frequencies and decay rates.

Despite these limitations, the simplicity of the linear framework is also a strength: it yields exact, interpretable results that serve as a rigorous foundation for tackling more complex, biologically realistic systems.

7 Conclusion

We have demonstrated that eigenvalue analysis provides a complete and interpretable characterisation of a two-dimensional linear neuronal model. The trace and determinant of the system matrix A serve as the minimal sufficient statistics for predicting whether the system will exhibit stable decay, damped oscillation, or unstable divergence. These predictions are validated by numerical simulation, and the stability boundaries are visualised in a trace-determinant map that makes parameter sensitivity immediately apparent.

The damped oscillatory regime, characterised by complex eigenvalues with negative real parts, reproduces key qualitative features of sub-threshold neuronal dynamics and provides the linear-algebraic intuition underlying the Hopf bifurcation in nonlinear models. The eigenvalue locus analysis further illustrates how a continuous parameter variation drives the system from stability to instability—a perspective with direct relevance to neuromorphic engineering and computational neuroscience.

All code, figures, and documentation are available at `https://github.com/calyxish/neuronal-stability-analysis`.

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